CCFU Proof 13

Parallelism Identity $\Pi(\ln\varphi) = \arctan(2)$

Given. Let $\varphi = (1+\sqrt{5})/2$, so $\varphi^2 = \varphi + 1$ and $\varphi > 1$.

Definition. The Lobachevsky parallelism function in $\mathbb{H}^2$ ($K = -1$):

$$\Pi(d) = 2\arctan(e^{-d}). \qquad \text{Equivalently: } \Pi(d) = \arccos(\tanh d).$$

Claim. $\Pi(\ln\varphi) = \arctan(2)$.

Proof via double angle formula.

$$\Pi(\ln\varphi) = 2\arctan(e^{-\ln\varphi}) = 2\arctan(1/\varphi).$$

Using $2\arctan(x) = \arctan\left(\frac{2x}{1-x^2}\right)$ when $x^2 < 1$:

$x = 1/\varphi$, $x^2 = 1/\varphi^2 < 1$. Since $x > 0$ and $x^2 < 1$, the result lies in $(0, \pi/2)$, so no branch correction is needed.

$$\frac{2x}{1-x^2} = \frac{2/\varphi}{1-1/\varphi^2} = \frac{2/\varphi}{(\varphi^2-1)/\varphi^2} = \frac{2\varphi}{\varphi^2-1} = \frac{2\varphi}{\varphi} = 2.$$

Therefore $\Pi(\ln\varphi) = 2\arctan(1/\varphi) = \arctan(2)$. ■

Verification via arccos **form.**

$$\tanh(\ln\varphi) = \frac{\varphi - 1/\varphi}{\varphi + 1/\varphi} = \frac{\varphi^2-1}{\varphi^2+1} = \frac{\varphi}{\varphi+2}.$$

Since $2\varphi - 1 = \sqrt{5}$:

$$\sqrt{5}\,\varphi = (2\varphi-1)\varphi = 2\varphi^2 - \varphi = 2(\varphi+1) - \varphi = \varphi + 2.$$

Therefore $\varphi/(\varphi+2) = \varphi/(\sqrt{5}\,\varphi) = 1/\sqrt{5}$, and

$$\arccos(1/\sqrt{5}) = \arctan(2). \qquad \checkmark$$

Both definitions give the same result. ■